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Hedging of the European option in discrete time under proportional transaction costs

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Abstract. In the paper hedging of the European option in a discrete time financial market with proportional transaction costs is studied. It is shown that for a certain class of options the set of portfolios which allow to hedge an option in a discrete time model with a bounded set of possible changes in a stock price is the same as the set of such portfolios, under assumption that the stock price evolution is given by a suitable CRR model.

Key words: European option, Self-financing strategy, Hedging, Transaction costs

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1 Introduction

In the paper we consider the problem of hedging of the European option in a discrete time market model.

Although the problem of hedging of contingent claims in discrete time under proportional transaction costs was studied in many papers (see [1]–[12]) it appears to be nontrivial to apply to the real market the results which were obtained for a general model (see [3], [4], [11]). From calculation point of view a so called Cox-Ross-Rubinstein (CRR) model is very convenient since in this particular model it is easy to get the exact value of the price of an option as well as the set of portfolios which allow to hedge the option (see for instance [1], [2], [9], [10]). On the other hand the Cox-Ross-Rubinstein model seems to be too simple to be a proper description of the real stock price evolution. In this paper however it is shown that for a special class of options which includes popular call option the set of portfolios which allow to hedge a

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contingent claim in a quite general model of the stock price process is the same as in the Cox-Ross-Rubinstein model. The result therefore seems to be interesting for practitioners since it justifies the use of the CRR model approach to price derivatives for a certain class of options. The possibility of reducing of the model of a stock price movement to the binomial model in case of the problem of hedging of the European option for the market with no transaction costs was considered in [5], [8] and [12].

2 The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, T a positive natural number and $\{\mathcal{F}_t, t = 0, \ldots, T\}$ a family of σ-algebras such that $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ for $t = 0, \ldots, T-1$ and $\mathcal{F}_T = \mathcal{F}$. We assume that Ω is finite (except Subsection 4.2).

Throughout this paper equalities and inequalities depending on $\omega \in \Omega$ if not stated otherwise hold for all $\omega \in \Omega$.

We consider a market with two assets, a risky stock and a riskless bond and assume that all assets are infinitely divisible.

The stock price movement is modelled by the process $\{S_t, t = 0, \ldots, T\}$ where S_t denotes the price of the stock at time t, for $t = 0, \ldots, T$. We assume that S_t is $\mathcal{F}_t$ measurable for $t = 0, \ldots, T$.

In our model the stock price process satisfies the following recursive formula:

$$S_{t+1} = (1 + \eta_{t+1})S_t, \quad t = 0, \ldots, T-1$$

where $S_0 > 0$ and $\{\eta_t\}_{t=1,\ldots,T}$ is a sequence of i.i.d. random variables such that $\eta_t \in \langle a, b \rangle$ where $-1 < a < 0$ and $b > 0$.

We assume that the following inequalities hold:

$$P(\eta_t = e) > 0 \quad \text{for } t = 1, \ldots, T \quad \text{and} \quad e \in \{a, b\}. \tag{2.1}$$

It is easily seen that the price of the stock is positive at each moment.

We assume that $\mathcal{F}_t = \sigma(\eta_u, 1 \le u \le t)$ for $t = 1, \ldots, T$.

For every $t = 0, \ldots, T-1$ and $\theta \in \langle a, b \rangle$ let S_t^θ be a random variable defined as follows:

$$S_t^\theta(\omega) = (1 + \theta)S_t(\omega) \quad \text{for } \omega \in \Omega.$$

We assume that the bond earns interest with a constant rate r such that $a > -1$ and $0 \le r < b$.

For transfers of wealth from one asset to another the proportional transaction costs are paid in our model. For every $t = 0, \ldots, T$ the cost of buying one share of the stock at time t is $(1 + \lambda)S_t$, where $\lambda \in [0, \infty)$, and the amount received for selling one share at time t is $(1 - \mu)S_t$, with $\mu \in [0, 1)$.

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a function defined as follows:

$$\rho(z) = \begin{cases} (1 + \lambda)z & \text{if } z \ge 0 \\ (1 - \mu)z & \text{if } z < 0 \end{cases}.$$

For any $(q_1, q_2) \in \mathbb{R}^2$ we define a set $C_{(q_1, q_2)}$ as follows:

$$C_{(q_1, q_2)} = \{(u, v) \in \mathbb{R}^2 \colon q_1 - u + \rho(q_2 - v) \le 0\}$$

A trading strategy (x, y) is a pair of processes $\{(x_t, y_t), t = 0, \ldots, T-1\}$ where x_t, y_t are $\mathcal{F}_t$ measurable for $t = 0, \ldots, T-1$. Here, x_t, y_t denote holdings (in cash) of bonds and shares of the stock respectively, held by the seller of the option at time t (after transaction at this moment) for $t = 0, \ldots, T-1$. Moreover, for a strategy (x, y) let $x_{-1}, y_{-1} \in \mathbb{R}$ denote respectively initial holdings (in cash) of bonds and shares of the stock in a portfolio of the seller of the option.

By convention we set $S_{-1} = S_0$.

A trading strategy (x, y) is said to be self-financing if:

$$x_0 - x_{-1} + \rho(y_0 - y_{-1}) \le 0$$

and

$$x_t - (1+r)x_{t-1} + \rho\left(y_t - \frac{S_t}{S_{t-1}} y_{t-1}\right) \le 0 \quad \text{for} \quad t = 1, \ldots, T-1$$

This means that at every trading moment, the sales must finance possible purchase.

Denote by $\mathcal{A}$ the set of all self-financing, trading strategies.

If $P(\eta_t = a) + P(\eta_t = b) = 1$ for $t = 1, \ldots, T$ and $0 < P(\eta_t = b) < 1$ for $t = 1, \ldots, T$ then we have so called Cox-Ross-Rubinstein model. We will denote such a model by $CRR(a, b)$.

3 Some auxiliary results

Throughout this paper functions if not stated otherwise are defined on $(0, \infty)$, measurable and take values in $\mathbb{R}$.

Let $p = (p_1, p_2)$ be a given pair of functions.

We define functions $c_{p,1}$ and $c_{p,2}$ as follows:

$$c_{p,1}(s) = \frac{p_1(s)}{1+\lambda} + p_2(s) \quad \text{and} \quad c_{p,2}(s) = \frac{p_1(s)}{1-\mu} + p_2(s) \quad \text{for all } s \in (0, \infty).$$

Throughout this paper equalities and inequalities depending on $s \in (0, \infty)$ if not stated otherwise hold for all $s \in (0, \infty)$.

Let the constant γ be defined as follows:

$$\gamma = (1+\lambda)(1+b) - (1-\mu)(1+a).$$

It is easy to check that $\gamma > 0$.

For simplicity of notation we write s^θ instead of $(1+\theta)s$ and $s^{\theta\delta}$ instead of $(1+\delta)(1+\theta)s$.

We define functions I_p^1 and I_p^2 as follows:

$$I_p^1(s) = (1-\mu)(b-a)c_{p,2}(s^a) - \gamma c_{p,1}(s^a) + (1+a)(\mu+\lambda)c_{p,1}(s^b)$$

and

$$I_p^2(s) = (1+\lambda)(b-a)c_{p,1}(s^b) - \gamma c_{p,2}(s^b) + (1+b)(\mu+\lambda)c_{p,2}(s^a).$$

Let Π denote the set of all pairs of functions p such that $I_p^1(s) \ge 0$ and $I_p^2(s) \ge 0$ for all $s \in \langle (1+a)^{T-1} S_0, (1+b)^{T-1} S_0 \rangle$.

Definition 3.1. *Let (p_1, p_2) be a given pair of functions. We define by induction pairs of functions $p^{(i)} = (p_1^{(i)}, p_2^{(i)})$ for $i \in \mathbb{N} \backslash \{0\}$ as follows:*

$$p_j^{(0)}(s) = p_j(s) \quad \textit{for } j = 1, 2$$

$$p_1^{(i)}(s) = \frac{(1-\mu)(1+\lambda)}{(1+r)\gamma}((1+b)c_{p^{(i-1)},2}(s^a) - (1+a)c_{p^{(i-1)},1}(s^b))$$

$$p_2^{(i)}(s) = \frac{-(1+r)}{(1+\lambda)(1+b)}p_1^{(i)}(s) + \frac{1}{1+b}c_{p^{(i-1)},1}(s^b).$$

For all $\theta \in \langle a,b\rangle$ we define functions $L_p^{1,\theta}$ and $L_p^{2,\theta}$ as follows:

$$\begin{aligned} L_p^{1,\theta}(s) = {} & \frac{1}{\gamma}((b-\theta)(1-\mu)c_{p,2}(s^a) + ((1+\lambda)(1+\theta) \\ & - (1-\mu)(1+a))c_{p,1}(s^b)) - c_{p,1}(s^\theta) \end{aligned}$$

and

$$\begin{aligned} L_p^{2,\theta}(s) = {} & \frac{1}{\gamma}(((1+\lambda)(1+b) - (1-\mu)(1+\theta))c_{p,2}(s^a) \\ & + (\theta-a)(1+\lambda)c_{p,1}(s^b)) - c_{p,2}(s^\theta). \end{aligned}$$

By Ψ we denote the set of all pairs of functions p such that $p \in \Pi$ and for all $\theta \in \langle a,b\rangle$ and $s \in \langle (1+a)^{T-1}S_0, (1+b)^{T-1}S_0\rangle$ the following inequalities are satisfied:

$$L_p^{1,\theta}(s) \geq 0$$
$$L_p^{2,\theta}(s) \geq 0$$
$$(b-r)L_p^{2,\theta}(s) - \frac{(\theta-r)}{\gamma}I_p^2(s) \geq 0$$
$$(r-a)L_p^{1,\theta}(s) + \frac{(\theta-r)}{\gamma}I_p^1(s) \geq 0.$$

For all $\theta \in \langle a,b\rangle$ and every $t = 0,\ldots,T-1$ let

$$\begin{aligned} G_p^\theta(t) = \Bigg\{ (x,y) \in \mathbb{R}^2\colon y \geq \max\Bigg\{ & \frac{-(1+r)}{(1+\lambda)(1+\theta)}x + \frac{1}{1+\theta}c_{p,1}(S_t^\theta), \\ & \frac{-(1+r)}{(1-\mu)(1+\theta)}x + \frac{1}{1+\theta}c_{p,2}(S_t^\theta)\Bigg\}\Bigg\}. \end{aligned}$$

Moreover, let $\widehat{G}_p(t) = G_p^a(t) \cap G_p^b(t)$ for $t = 0,\ldots,T-1$.

It is clear that for all $\theta \in \langle a,b\rangle$ and $t = 0,\ldots,T-1$ the set $G_p^\theta(t)$ depends on $\omega \in \Omega$ and consequently, the same holds for $\widehat{G}_p(t)$.

Lemma 3.2. *For all $\omega \in \Omega, \theta \in \langle a,b\rangle$ and $t = 0,\ldots,T-1$ we have the following equivalence:*

$$\hat{G}_p(t)(\omega) \subseteq G_p^\theta(t)(\omega) \quad \textit{if and only if } L_p^{1,\theta}(S_t(\omega)) \geq 0 \quad \textit{and} \quad L_p^{2,\theta}(S_t(\omega)) \geq 0.$$

Proof. We fix $\omega \in \Omega, \theta \in \langle a,b\rangle$ and $t \in \{0,\ldots,T-1\}$ in this proof. Moreover, throughout this proof we omit the fixed ω in the notation.

Let,

$$E_1^\theta = \left\{(x,y) \in \mathbb{R}^2 : y \geq \frac{-(1+r)}{(1+\lambda)(1+\theta)}x + \frac{1}{1+\theta}c_{p,1}(S_t^\theta)\right\} \quad \text{and}$$
$$E_2^\theta = \left\{(x,y) \in \mathbb{R}^2 : y \geq \frac{-(1+r)}{(1-\mu)(1+\theta)}x + \frac{1}{1+\theta}c_{p,2}(S_t^\theta)\right\}.$$

It is easy to see that $\hat{G}_p(t) \subseteq G_p^\theta(t)$ if and only if $p^{(1)}(S_t) \in E_1^\theta \cap E_2^\theta$.

Then by direct calculation, we obtain for $i = 1, 2$ that $p^{(1)}(S_t) \in E_i^\theta$ if and only if $L_p^{i,\theta}(S_t) \geq 0$, which completes the proof. □

For simplicity of notation let

$$A_1 = \frac{(1+\lambda)(1+r) - (1-\mu)(1+a)}{(1+r)\gamma}, \quad B_1 = \frac{(b-r)(1-\mu)}{(1+r)\gamma}$$

$$A_2 = \frac{(r-a)(1+\lambda)}{(1+r)\gamma}, \quad B_2 = \frac{(1+\lambda)(1+b) - (1-\mu)(1+r)}{(1+r)\gamma}.$$

Moreover, for $\theta \in \langle a, b \rangle$ let

$$\alpha_1(\theta) = \frac{(1+\lambda)(1+\theta) - (1-\mu)(1+a)}{\gamma}, \quad \beta_1(\theta) = \frac{(b-\theta)(1-\mu)}{\gamma}$$

$$\alpha_2(\theta) = \frac{(\theta-a)(1+\lambda)}{\gamma}, \quad \beta_2(\theta) = \frac{(1+\lambda)(1+b) - (1-\mu)(1+\theta)}{\gamma}.$$

By a standard calculation we have the following lemma:

Lemma 3.3. *For* $i \in \mathbb{N}\backslash\{0\}$ *we have* $c_{p^{(i)},1}(s) = A_1 c_{p^{(i-1)},1}(s^b) + B_1 c_{p^{(i-1)},2}(s^a)$ *and* $c_{p^{(i)},2}(s) = A_2 c_{p^{(i-1)},1}(s^b) + B_2 c_{p^{(i-1)},2}(s^a)$.

By Lemma 3.3 we obtain the following useful identities:

Lemma 3.4. *For all* $\theta \in \langle a, b \rangle, s \in (0, \infty)$ *and* $i \in \mathbb{N}\backslash\{0\}$ *we have* $L_{p^{(i)}}^{1,\theta}(s) = A_1 L_{p^{(i-1)}}^{1,\theta}(s^b) + B_1 L_{p^{(i-1)}}^{2,\theta}(s^a) - \frac{(1-\mu)(\theta-r)}{\gamma^2(1+r)} I_{p^{(i-1)}}^2(s^a)$ and $L_{p^{(i)}}^{2,\theta}(s) = A_2 L_{p^{(i-1)}}^{1,\theta}(s^b) + B_2 L_{p^{(i-1)}}^{2,\theta}(s^a) + \frac{(1+\lambda)(\theta-r)}{\gamma^2(1+r)} I_{p^{(i-1)}}^1(s^b)$.

Proof. We fix $\theta \in \langle a, b \rangle$ and $i \in \mathbb{N}\backslash\{0\}$ in this proof.

Notice that:

$$\alpha_1 B_1 - A_1 \beta_1(\theta) = \frac{(1-\mu)(\theta-r)}{(1+r)\gamma} \tag{3.1}$$

$$\beta_1(\theta) A_2 - B_1 \alpha_2(\theta) = \frac{(1-\mu)(1+\lambda)(a-b)(\theta-r)}{(1+r)\gamma^2} \tag{3.2}$$

$$\beta_1(\theta) B_2 - B_1 \beta_2(\theta) = \frac{(1-\mu)(1+b)(\mu+\lambda)(r-\theta)}{(1+r)\gamma^2}. \tag{3.3}$$

Moreover, $L^{1,\theta}_{p^{(i)}}(s) = \alpha_1(\theta)c_{p^{(i)},1}(s^b) + \beta_1(\theta)c_{p^{(i)},2}(s^a) - c_{p^{(i)},1}(s^\theta)$.

From Lemma 3.3 we have:

$$L^{1,\theta}_{p^{(i)}}(s) = \alpha_1(\theta)(A_1c_{p^{(i-1)},1}(s^{bb}) + B_1c_{p^{(i-1)}.2}(s^{ba})) + \beta_1(\theta)(A_2c_{p^{(i-1)}.1}(s^{ab}) + B_2c_{p^{(i-1)},2}(s^{aa})) - (A_1c_{p^{(i-1)},1}(s^{\theta b}) + B_1c_{p^{(i-1)},2}(s^{\theta a})).$$

Therefore, from (3.1), (3.2), (3.3) we obtain:

$$\begin{aligned} L^{1,\theta}_{p^{(i)}}(s) &= A_1(\alpha_1(\theta)c_{p^{(i-1)},1}(s^{bb}) + \beta_1(\theta)c_{p^{(i-1)},2}(s^{ba}) - c_{p^{(i-1)},1}(s^{\theta b})) \\ &\quad + B_1(\alpha_2(\theta)c_{p^{(i-1)},1}(s^{ab}) + \beta_2(\theta)c_{p^{(i-1)},2}(s^{aa}) - c_{p^{(i-1)},2}(s^{\theta a})) \\ &\quad + \frac{(1-\mu)(\theta-r)}{(1+r)\gamma^2}(\gamma c_{p^{(i-1)},2}(s^{ba}) + (1+\lambda)(a-b)c_{p^{(i-1)},1}(s^{ab}) \\ &\quad - (1+b)(\mu+\lambda)c_{p^{(i-1)},2}(s^{aa})). \end{aligned}$$

Consequently,

$$L^{1,\theta}_{p^{(i)}}(s) = A_1L^{1,\theta}_{p^{(i-1)}}(s^b) + B_1L^{2,\theta}_{p^{(i-1)}}(s^a) - \frac{(1-\mu)(\theta-r)}{(1+r)\gamma^2}I^2_{p^{(i-1)}}(s^a).$$

The proof of the second identity is similar. Namely, we have:

$$\alpha_2(\theta)A_1 - \alpha_1A_2 = \frac{(\lambda+\mu)(1+a)(\theta-r)(1+\lambda)}{(1+r)\gamma^2} \tag{3.4}$$

$$\alpha_2(\theta)B_1 - A_2\beta_1(\theta) = \frac{(1+\lambda)(1-\mu)(b-a)(\theta-r)}{(1+r)\gamma^2} \tag{3.5}$$

$$\beta_2(\theta)A_2 - B_2\alpha_2(\theta) = \frac{(1+\lambda)(r-\theta)}{(1+r)\gamma}. \tag{3.6}$$

Moreover, $L^{2,\theta}_{p^{(i)}}(s) = \alpha_2(\theta)c_{p^{(i)},1}(s^b) + \beta_2(\theta)c_{p^{(i)},2}(s^a) - c_{p^{(i)},2}(s^\theta)$.

From Lemma 3.3 we obtain:

$$L^{2,\theta}_{p^{(i)}}(s) = \alpha_2(\theta)(A_1c_{p^{(i-1)},1}(s^{bb}) + B_1c_{p^{(i-1)},2}(s^{ba})) + \beta_2(\theta)(A_2c_{p^{(i-1)},1}(s^{ab}) + B_2c_{p^{(i-1)},2}(s^{aa})) - (A_2c_{p^{(i-1)},1}(s^{\theta b}) + B_2c_{p^{(i-1)},2}(s^{\theta a})).$$

Using (3.4), (3.5), (3.6) we have:

$$\begin{aligned} L^{2,\theta}_{p^{(i)}}(s) &= A_2(\alpha_1(\theta)c_{p^{(i-1)},1}(s^{bb}) + \beta_1(\theta)c_{p^{(i-1)},2}(s^{ba}) - c_{p^{(i-1)},1}(s^{\theta b})) \\ &\quad + B_2(\alpha_2(\theta)c_{p^{(i-1)},1}(s^{ab}) + \beta_2(\theta)c_{p^{(i-1)},2}(s^{aa}) - c_{p^{(i-1)},2}(s^{\theta a})) \\ &\quad + \frac{(1+\lambda)(\theta-r)}{(1+r)\gamma^2}((\lambda+\mu)(1+a)c_{p^{(i-1)},1}(s^{bb}) \\ &\quad +(1-\mu)(b-a)c_{p^{(i-1)},2}(s^{ba}) - \gamma c_{p^{(i-1)},1}(s^{ab})). \end{aligned}$$

Consequently, $L^{2,\theta}_{p^{(i)}}(s) = A_2L^{1,\theta}_{p^{(i-1)}}(s^b) + B_2L^{2,\theta}_{p^{(i-1)}}(s^a) + \frac{(1+\lambda)(\theta-r)}{(1+r)\gamma^2}I^1_{p^{(i-1)}}(s^b)$ and the proof is completed. □

From the proof of Lemma 3 in [6] we have the following fact:

Lemma 3.5. *For $i \in N\backslash\{0\}$ the functions $I^1_{p^{(i)}}$ and $I^2_{p^{(i)}}$ satisfy the following recursive identities:*

$$I^1_{p^{(i)}}(s) = A_1 I^1_{p^{(i-1)}}(s^b) + \frac{(1-\mu)}{(1+\lambda)} A_2 I^2_{p^{(i-1)}}(s^a)$$

and

$$I^2_{p^{(i)}}(s) = \frac{(1+\lambda)}{(1-\mu)} B_1 I^1_{p^{(i-1)}}(s^b) + B_2 I^2_{p^{(i-1)}}(s^a).$$

Concluding our technical results, we obtain:

Theorem 3.6. *Let p be a pair of functions such that $p \in \Psi$. Then $L^{j,\theta}_{p^{(i)}}(S_{T-i-1}) \geq 0$ for all $\omega \in \Omega, \theta \in \langle a, b\rangle, j \in \{1,2\}$ and $i = 0, \ldots, T-1$.*

Proof: Let $\theta \in \langle a, b\rangle$ be fixed in this proof.

We use a backward induction.

Let p be a given pair of functions. It is clear that,

$$L^{1,\theta}_{p^{(0)}}(S_{T-1}) \geq 0$$

$$L^{2,\theta}_{p^{(0)}}(S_{T-1}) \geq 0$$

$$(b-r)L^{2,\theta}_{p^{(0)}}(S_{T-1}) - \frac{(\theta-r)}{\gamma} I^2_{p^{(0)}}(S_{T-1}) \geq 0$$

$$(r-a)L^{1,\theta}_{p^{(0)}}(S_{T-1}) + \frac{(\theta-r)}{\gamma} I^1_{p^{(0)}}(S_{T-1}) \geq 0$$

Assume that for some $i \in \{0, \ldots, T-2\}$ we have

$$L^{1,\theta}_{p^{(i)}}(S_{T-i-1}) \geq 0$$

$$L^{2,\theta}_{p^{(i)}}(S_{T-i-1}) \geq 0$$

$$(b-r)L^{2,\theta}_{p^{(i)}}(S_{T-i-1}) - \frac{(\theta-r)}{\gamma} I^2_{p^{(i)}}(S_{T-i-1}) \geq 0$$

$$(r-a)L^{1,\theta}_{p^{(i)}}(S_{T-i-1}) + \frac{(\theta-r)}{\gamma} I^1_{p^{(i)}}(S_{T-i-1}) \geq 0.$$

We shall prove first that $L^{1,\theta}_{p^{(i+1)}}(S_{T-i-2}) \geq 0$ and $L^{2,\theta}_{p^{(i+1)}}(S_{T-i-2}) \geq 0$.

From the inequality $(b-r)L^{2,\theta}_{p^{(i)}}(S_{T-i-1}) - \frac{(\theta-r)}{\gamma} I^2_{p^{(i)}}(S_{T-i-1}) \geq 0$ and (2.1) we have $(b-r)L^{2,\theta}_{p^{(i)}}(S^a_{T-i-2}) - \frac{(\theta-r)}{\gamma} I^2_{p^{(i)}}(S^a_{T-i-2}) \geq 0$.

Multiplying both sides of the last inequality by $\frac{(1-\mu)}{(1+r)\gamma}$ we obtain:

$$B_1 L^{2,\theta}_{p^{(i)}}(S^a_{T-i-2}) - \frac{(1-\mu)(\theta-r)}{\gamma^2(1+r)} I^2_{p^{(i)}}(S^a_{T-i-2}) \geq 0.$$

By the inequality $L^{1,\theta}_{p^{(i)}}(S_{T-i-1}) \geq 0$ and (2.1) we have $A_1 L^{1,\theta}_{p^{(i)}}(S^b_{T-i-2}) \geq 0$. Consequently, $A_1 L^{1,\theta}_{p^{(i)}}(S^b_{T-i-2}) + B_1 L^{2,\theta}_{p^{(i)}}(S^a_{T-i-2}) - \frac{(1-\mu)(\theta-r)}{\gamma^2(1+r)} I^2_{p^{(i)}}(S^a_{T-i-2}) \geq 0$.

Therefore using Lemma 3.4 we get $L^{1,\theta}_{p^{(i+1)}}(S_{T-i-2}) \geq 0$.

Now, from the inequality $(r-a)L^{1,\theta}_{p^{(i)}}(S_{T-i-1})+\frac{(\theta-r)}{\gamma}I^{1}_{p^{(i)}}(S_{T-i-1})\geq 0$ and (2.1) we have $(r-a)L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+\frac{(\theta-r)}{\gamma}I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\geq 0$.

Multiplying both sides of the last inequality by $\frac{(1+\lambda)}{(1+r)\gamma}$ we obtain:

$$A_2L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+\frac{(1+\lambda)(\theta-r)}{\gamma^2(1+r)}I^{1}_{p^{(i)}}(S^{b}_{t-i-2})\geq 0.$$

By the inequality $L^{2,\theta}_{p^{(i)}}(S_{T-i-1})\geq 0$ and (2.1) we have $B_2L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})\geq 0$. Consequently, $A_2L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+B_2L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})+\frac{(1+\lambda)(\theta-r)}{\gamma^2(1+r)}I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\geq 0$.

Therefore using Lemma 3.4 we get $L^{2,\theta}_{p^{(i+1)}}(S_{T-i-2})\geq 0$.

We shall prove now that $(b-r)L^{2,\theta}_{p^{(i+1)}}(S_{T-i-2})-\frac{(\theta-r)}{\gamma}I^{2}_{p^{(i+1)}}(S_{T-i-2})\geq 0$ and $(r-a)L^{1,\theta}_{p^{(i+1)}}(S_{T-i-2})+\frac{(\theta-r)}{\gamma}I^{1}_{p^{(i+1)}}(S_{T-i-2})\geq 0$.

From the inequality $(b-r)L^{2,\theta}_{p^{(i)}}(S_{T-i-1})-\frac{(\theta-r)}{\gamma}I^{2}_{p^{(i)}}(S_{T-i-1})\geq 0$ and (2.1) we have $(b-r)B_2L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})-\frac{(\theta-r)}{\gamma}B_2I^{2}_{p^{(i)}}(S^{a}_{T-i-2})\geq 0$.

By the inequality $L^{1,\theta}_{p^{(i)}}(S_{T-i-1})\geq 0$ and (2.1) we have $A_2L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})\geq 0$ and therefore $(b-r)(A_2L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+B_2L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2}))-\frac{(\theta-r)}{\gamma}B_2I^{2}_{p^{(i)}}(S^{a}_{T-i-2})\geq 0$.

From the identity $(b-r)\frac{(1+\lambda)(\theta-r)}{\gamma^2(1+r)}=\frac{(\theta-r)}{\gamma}\frac{(1+\lambda)}{(1-\mu)}B_1$ we have

$$(b-r)\left(A_2L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+B_2L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})+\frac{(1+\lambda)(\theta-r)}{\gamma^2(1+r)}I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\right)$$
$$-\frac{(\theta-r)}{\gamma}\left(\frac{(1+\lambda)}{(1-\mu)}B_1I^{1}_{p^{(i)}}(S^{b}_{T-i-2})+B_2I^{2}_{p^{(i)}}(S^{a}_{T-i-2})\right)\geq 0.$$

Consequently, using Lemmas 3.4 and 3.5 we get:

$$(b-r)L^{2,\theta}_{p^{(i+1)}}(S_{T-i-2})-\frac{(\theta-r)}{\gamma}I^{2}_{p^{(i+1)}}(S_{T-i-2})\geq 0.$$

Starting now from the inequality $(r-a)L^{1,\theta}_{p^{(i)}}(S_{T-i-1})+\frac{(\theta-r)}{\gamma}I^{1}_{p^{(i)}}(S_{T-i-1})\geq 0$ by (2.1) we obtain $(r-a)A_1L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+\frac{(\theta-r)}{\gamma}A_1I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\geq 0$.

By the inequality $L^{2,\theta}_{p^{(i)}}(S_{T-i-1})\geq 0$ and (2.1) we have $B_1L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})\geq 0$. Consequently, $(r-a)(A_1L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+B_1L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2}))+\frac{(\theta-r)}{\gamma}A_1I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\geq 0$. Thus, using the identity $(r-a)\frac{(1-\mu)(\theta-r)}{\gamma^2(1+r)}=\frac{(\theta-r)}{\gamma}\frac{(1-\mu)}{(1+\lambda)}A_2$ we obtain:

$$(r-a)\left(A_1L^{1,\theta}_{p^{(i)}}(S^{b}_{T-i-2})+B_1L^{2,\theta}_{p^{(i)}}(S^{a}_{T-i-2})-\frac{(1-\mu)(\theta-r)}{\gamma^2(1+r)}I^{2}_{p^{(i)}}(S^{a}_{T-i-2})\right)$$
$$+\frac{(\theta-r)}{\gamma}\left(\frac{(1-\mu)}{(1+\lambda)}A_2I^{2}_{p^{(i)}}(S^{a}_{T-i-2})+A_1I^{1}_{p^{(i)}}(S^{b}_{T-i-2})\right)\geq 0.$$

Consequently, using Lemmas 3.4 and 3.5 we get:

$$(r-a)L^{1,\theta}_{p^{(i+1)}}(S_{T-i-2})+\frac{(\theta-r)}{\gamma}I^{1}_{p^{(i+1)}}(S_{T-i-2})\geq 0.$$

The proof by backward induction is therefore completed. □

To study examples of options we need the following fact:

Remark 3.7. *For any pair of functions* $p = (p_1, p_2)$ *and all* $s \in (0, \infty)$ *the following identities hold:*

$$L_p^{1,a}(s) = \frac{1}{\gamma} I_p^1(s), \quad L_p^{1,b}(s) = 0,$$

$$L_p^{2,a}(s) = 0, \quad L_p^{2,b}(s) = \frac{1}{\gamma} I_p^2(s),$$

4 Hedging of the option

Let $\varphi = (\varphi_1, \varphi_2)$ be a given pair of functions. An option is a pair $(\varphi_1(S_T), \varphi_2(S_T))$ of random variables where $\varphi_1(S_T), \varphi_2(S_T)$ denote the amounts of bonds and shares of the stock (in cash) respectively, that are paid at time T to the option's holder assuming that the holder of the option exercises his claim. Throughout the paper we identify an option with the pay-off pair of functions φ.

An option in this paper will be also called a contingent claim or a European option since the option's holder can get his payment only at time T.

It can be easily seen that for any option φ there exists a unique pair of functions $f_\varphi = (f_{\varphi,1}, f_{\varphi,2})$ satisfying:

$$C_{(f_{\varphi,1}(s), f_{\varphi,2}(s))} = C_{(0,0)} \cap C_{(\varphi_1(s), \varphi_2(s))} \text{ for } s \in (0, \infty) \tag{4.1}$$

and

$$f_{\varphi,1}(s) = \varphi_1(s) \mathbf{1}_{\{\varphi_1(s) + \varphi_2(s) > 0\}} \text{ if } \lambda + \mu = 0.$$

Any strategy $(x, y) \in \mathcal{A}$ of the option's seller in order to assure the payment of the holder of the option φ has to satisfy the inequality:

$$\varphi_1(S_T) - (1 + r)x_{T-1} + \rho\left(\varphi_2(S_T) - \frac{S_T}{S_{T-1}} y_{T-1}\right) \le 0. \tag{4.2}$$

Moreover, such a strategy also has to satisfy the inequality:

$$\rho\left(-\frac{S_T}{S_{T-1}} y_{T-1}\right) \le (1 + r)x_{T-1} \tag{4.3}$$

which implies that the seller of the contingent claim can reach simultaneously 0 in the number of bonds and shares of the stock. In other words (4.3) means that at time T the seller of the option can pay all his debts.

We say that a trading strategy $(x, y) \in \mathcal{A}$ hedges a contingent claim $\varphi = (\varphi_1, \varphi_2)$ (or is a hedging against φ) if:

$$f_{\varphi,1}(S_T) - (1 + r)x_{T-1} + \rho\left(f_{\varphi,2}(S_T) - \frac{S_T}{S_{T-1}} y_{T-1}\right) \le 0. \tag{4.4}$$

The last inequality is equivalent to the simultaneous holding of (4.2) and (4.3).

The inequality (4.4) implies that a trading strategy $(x, y) \in \mathcal{A}$ hedges an option φ if and only if it is possible using this strategy to assure the payments $f_{\varphi,1}(S_T), f_{\varphi,2}(S_T)$ in bonds and shares of the stock (in cash) respectively, at time T.

For all $\omega \in \Omega$ and $t = 0, ..., T-1$ we define a set $H_\varphi(t)(\omega)$ as follows:

$H_\varphi(t)(\omega) = \{(u, v) \in \mathbb{R}^2 :$ there exists $(x, y) \in \mathcal{A}$ such that $(x_{t-1}, y_{t-1})(\omega) = (u, v)$ and $P[f_{\varphi,1}(S_T) - (1+r)x_{T-1} + \rho(f_{\varphi,2}(S_T) - \frac{S_T}{S_{T-1}} y_{T-1}) \leq 0 \mid \mathcal{F}_t](\omega) = 1\}$.

$H_\varphi(t)$ is a set of pre-transaction portfolios which at time t guarantee hedging of the option φ at time T for every $t = 0, ..., T-1$.

Moreover, let $H_\varphi(T) = C_{(f_{\varphi,1}(S_T), f_{\varphi,2}(S_T))}$. It is clear that $H_\varphi(T)$ is a set of pre-transaction portfolios which at time T guarantee the payments $f_{\varphi,1}(S_T), f_{\varphi,2}(S_T)$ in bonds and shares of the stock (in cash) respectively, at time T.

For every $t = 0, ..., T-1$ let $H_\varphi^{CRR}(t)$ be defined in the same way as $H_\varphi(t)$ assuming additionally that $P(\eta_u = a) + P(\eta_u = b) = 1$ and $0 < P(\eta_u = a) < 1$ for $u = t+1, ..., T$.

$H_\varphi^{CRR}(t)$ is a set of pre-transaction portfolios which at time t guarantee hedging of the contingent claim φ at time T if the stock price movement from the moment t until time T is the same as in the $CRR(a, b)$ model.

The seller's price of a contingent claim φ is defined by:

$$\pi(\varphi) = \inf\{x_0 + \rho(y_0), (x, y) \in \mathcal{A} \text{ and hedges } \varphi\}$$

It is easily seen that $H_\varphi(0)$ does not depend on $\omega \in \Omega$ and we have the equality $\pi(\varphi) = \inf\{x \in \mathbb{R}, (x, 0) \in H_\varphi(0)\}$.

For every $i = 0, ..., T$ let $f_\varphi^{(i)} = (f_{\varphi,1}^{(i)}, f_{\varphi,2}^{(i)})$ denote a pair of functions which is a result of the i-th iteration of the operator from Definition 3.1 on the pair of functions $(f_{\varphi,1}, f_{\varphi,2})$.

By Theorem 1 of [10] we have the following fact:

Theorem 4.1. *Let φ be an option such that $f_\varphi \in \Pi$.*

Then $H_\varphi^{CRR}(t) = C_{(f_{\varphi,1}^{(T-t)}(S_T), f_{\varphi,2}^{(T-t)}(S_T))}$ for all $\omega \in \Omega$ and $t = 0, ..., T-1$.

The main result is:

Theorem 4.2. *Let φ be an option such that $f_\varphi \in \Psi$.*

Then $H_\varphi(t) = H_\varphi^{CRR}(t)$ for all $\omega \in \Omega$ and $t = 0, ..., T-1$.

Proof. From Theorem 3.6 we have $L^{1,\theta}_{f_\varphi^{(T-t-1)}}(S_T) \geq 0$ and $L^{2,\theta}_{f_\varphi^{(T-t-1)}}(S_T) \geq 0$ for all $\omega \in \Omega, \theta \in \langle a, b\rangle$ and $t = 0, ..., T-1$.

Thus from Lemma 3.2 we have $\widehat{G}_{f_\varphi^{(T-t-1)}}(t) \subseteq \bigcap_{\theta \in \langle a,b\rangle} G^\theta_{f_\varphi^{(T-t-1)}}(t)$ for all $\omega \in \Omega$ and $t = 0, ..., T-1$. It is easy to verify that $f_\varphi^{(T-t)}(S_T) \in \widehat{G}_{f_\varphi^{(T-t-1)}}(t)$ for all $\omega \in \Omega$ and $t = 0, ..., T-1$. Consequently, we get:

$$f_\varphi^{(T-t)}(S_T) \in \bigcap_{\theta \in \langle a,b\rangle} G^\theta_{f_\varphi^{(T-t-1)}}(t) \text{ for all } \omega \in \Omega \text{ and} t = 0, ..., T-1. \tag{4.5}$$

We use now a backward induction.

It is clear that $H_\varphi(T) = C_{(f_{\varphi,1}^{(0)}(S_T), f_{\varphi,2}^{(0)}(S_T))}$.

Assume that for some $t = 1, \ldots, T-1$ we have:

$$H_\varphi(t+1) = C_{(f_{\varphi,1}^{(T-t-1)}(S_{t+1}), f_{\varphi,2}^{(T-t-1)}(S_{t+1}))}.$$

Then, it is not difficult to check that $\bigcap_{\theta \in \langle a,b \rangle} G^\theta_{f_\varphi^{(T-t-1)}}(t) \subseteq H_\varphi(t)$ for all $\omega \in \Omega$.

Consequently, from (4.5) we get $f_\varphi^{(T-t)}(S_T) \in H_\varphi(t)$ for all $\omega \in \Omega$ and therefore it is easy to show that $C_{(f_{\varphi,1}^{(T-t)}(S_T), f_{\varphi,2}^{(T-t)}(S_T))} \subseteq H_\varphi(t)$ for all $\omega \in \Omega$. Thus, by Theorem 4.1 we obtain $H_\varphi^{CRR}(t) \subseteq H_\varphi(t)$ for all $\omega \in \Omega$.

From (2.1) we have $H_\varphi(t) \subseteq H_\varphi^{CRR}(t)$ for all $\omega \in \Omega$. Consequently, we obtain $H_\varphi(t) = H_\varphi^{CRR}(t)$.

By backward induction we have $H_\varphi(t) = H_\varphi^{CRR}(t)$ for every $t = 0, \ldots, T-1$ and the proof is therefore completed. □

Remark 4.3. *If* $\mu = \lambda = 0$ *then* $f_\varphi \in \Psi$ *if and only if* $\frac{b-\theta}{b-a}(\varphi_1(s^a) + \varphi_2(s^a)) + \frac{\theta-a}{b-a}(\varphi_1(s^b) + \varphi_2(s^b)) - (\varphi_1(s^\theta) + \varphi_2(s^\theta)) \geq 0$ *for all* $\omega \in \Omega, \theta \in \langle a, b \rangle$ *and* $s \in \langle (1+a)^{T-1} S_0, (1+b)^{T-1} S_0 \rangle$.

Remark 4.4. *If we additionally assume that* $P(\eta_t = 0) > 0$ for $t = 1, \ldots, T$ *then a set of pre-transaction portfolios that at a given moment guarantee hedging of the option* φ *such that* $f_\varphi \in \Psi$ *is the same as the analogous set for the American version of this option (pricing of the American option is considered e.g. in [7]).*

4.1 Examples

The following fact is useful to check that for the options described later Theorem 4.2 holds:

Lemma 4.5. *Let* $\delta \in \langle a, b \rangle$ *and let h be a continuous function defined on* $\langle a, b \rangle$ *as follows:*

$$h(\theta) = \begin{cases} \alpha_1 \theta + \beta_1 & \text{if } \theta \in \langle a, \delta \rangle \\ \alpha_2 \theta + \beta_2 & \text{if } \theta \in (\delta, b \rangle \end{cases}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ *for* $i \in \{1, 2\}$ *and* $\alpha_1 > \alpha_2$.

Then, $\inf_{\theta \in \langle a,b \rangle} h(\theta) \geq 0$ if and only if $\min\{h(a), h(b)\} \geq 0$.

4.1.1 The European call option with delivery

We assume in this example that $\mu + \lambda > 0$.

The holder of the option has the right to buy one unit of the stock for the price K at time T.

We have $\varphi_1(s) = -K$ and $\varphi_2(s) = s$.

The pair f_φ is given as follows:

$$f_{\varphi,1}(s) = \begin{cases} -K & \text{if } s \geq \frac{K}{1-\mu} \\ \frac{1-\mu}{\lambda+\mu}(K-(1+\lambda)s) & \text{if } \frac{K}{1+\lambda} \leq s < \frac{K}{1-\mu} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\varphi,2}(s) = \begin{cases} s & \text{if } s \geq \frac{K}{1-\mu} \\ \frac{1+\lambda}{\lambda+\mu}s - \frac{K}{\lambda+\mu} & \text{if } \frac{K}{1+\lambda} \leq s < \frac{K}{1-\mu} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to calculate that $c_{f_\varphi,1}(s) = (s - \frac{K}{1+\lambda})^+$ and $c_{f_\varphi,2}(s) = (s - \frac{K}{1-\mu})^+$. Moreover, it is not difficult to prove the following fact:

Proposition 4.6. *For the European call option with delivery we have* $f_\varphi \in \Pi$.

For all $s \in (0,\infty)$ let $h^{1,s}_{f_\varphi}$ and $h^{2,s}_{f_\varphi}$ be measurable and taking values in $\mathbb{R}$ functions defined on $\langle a,b\rangle$ as follows:

$h^{i,s}_{f_\varphi}(\theta) = L^{i,\theta}_{f_\varphi}(s)$ for all $\theta \in \langle a,b\rangle$ and $i \in \{1,2\}$

Let $s \in \left\langle (1+a)^{T-1}S_0, (1+b)^{T-1}S_0 \right\rangle$. It is easy to check that the functions $h^{1,s}_{f_\varphi}$ and $h^{2,s}_{f_\varphi}$ satisfy the assumptions of Lemma 4.5. Moreover, from Remark 3.7 and Proposition 4.6 we have $h^{i,s}_{f_\varphi}(a) \geq 0$ and $h^{i,s}_{f_\varphi}(b) \geq 0$ for $i \in \{1,2\}$. Consequently, from Lemma 4.5 we have $L^{1,\theta}_{f_\varphi}(s) \geq 0$ and $L^{2,\theta}_{f_\varphi}(s) \geq 0$ for all $\theta \in \langle a,b\rangle$.

For all $s \in (0,\infty)$ let $\widehat{h}^{1,s}_{f_\varphi}$ and $\widehat{h}^{2,s}_{f_\varphi}$ be measurable and taking values in $\mathbb{R}$ functions defined on $\langle a,b\rangle$ as follows:

$$\widehat{h}^{1,s}_{f_\varphi}(\theta) = (r-a)L^{1,\theta}_{f_\varphi}(s) + \frac{(\theta-r)}{\gamma}I^1_{f_\varphi}(s)$$

and

$$\widehat{h}^{2,s}_{f_\varphi}(\theta) = (b-r)L^{2,\theta}_{f_\varphi}(s) - \frac{(\theta-r)}{\gamma}I^2_{f_\varphi}(s)$$

for all $\theta \in \langle a,b\rangle$.

Let $s \in \langle (1+a)^{T-1}S_0, (1+b)^{T-1}S_0\rangle$. It is easy to notice that the functions $\widehat{h}^{i,s}_{f_\varphi}$ for $i = 1,2$ satisfy the assumptions of Lemma 4.5. Moreover, from Remark 3.7 and Proposition 4.6 we have $\widehat{h}^{i,s}_{f_\varphi}(a) \geq 0$ and $\widehat{h}^{i,s}_{f_\varphi}(b) \geq 0$ for $i \in \{1,2\}$. Therefore from Lemma 4.5 we have $(b-r)L^{2,\theta}_{f_\varphi}(s) - \frac{(\theta-r)}{\gamma}I^2_{f_\varphi}(s) \geq 0$ and $(r-a)L^{1,\theta}_{f_\varphi}(s) + \frac{(\theta-r)}{\gamma}I^1_{f_\varphi}(s) \geq 0$ for all $\theta \in \langle a,b\rangle$.

Consequently, by Theorem 4.2 for the European call option with delivery we obtain $H_\varphi(t) = H^{CRR}_\varphi(t)$ for $t = 0,...,T-1$.

4.1.2 The European call option with cash settlement:

We have $\varphi_1(s) = (s-K)^+$ *and* $\varphi_2(s) = 0$.

It is easy to see that $\varphi = f_\varphi$.

We assume in this example that $(1-\mu)(1+b) > (1+\lambda)(1+r)$.

It is not difficult to check that $H_\varphi(T-1) = C_{(g_1(S_{T-1}),g_2(S_{T-1}))}$ where g_1 and g_2 are the functions defined as follows:

$$g_1(s) = \begin{cases} \frac{-K}{(1+r)} & \text{if } s \geq \frac{K}{1+a} \\ \frac{((1+b)s-K)}{(a-b)(1+r)}(1+a) & \text{if } \frac{K}{1+b} \leq s < \frac{K}{1+a} \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(s) = \begin{cases} \frac{s}{1-\mu} & \text{if } s \geq \frac{K}{1+a} \\ \frac{((1+b)s-K)}{(1-\mu)(b-a)} & \text{if } \frac{K}{1+b} \leq s < \frac{K}{1+a} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that $H_\varphi(T-1) = H_\varphi^{CRR}(T-1)$.
Let $g = (g_1, g_2)$. It is not difficult to prove the following fact:

Proposition 4.7. *For the pair of functions $g = (g_1, g_2)$ defined above we have $g \in \Pi$.*

Stepping in the same manner as in the case of the European call option with delivery with Proposition 4.6 replaced with Proposition 4.7 we get:

$$L_g^{1,\theta}(s) \geq 0$$

$$L_g^{2,\theta}(s) \geq 0$$

$$(b-r)L_g^{2,\theta}(s) - \frac{(\theta - r)}{\gamma} I_g^2(s) \geq 0$$

$$(r-a)L_g^{1,\theta}(s) + \frac{(\theta - r)}{\gamma} I_g^1(s) \geq 0$$

for all $\theta \in \langle a, b\rangle$ *and* $s \in \langle (1+a)^{T-1}S_0, (1+b)^{T-1}S_0\rangle$.
Therefore, by Theorem 4.2 with a new last time $T-1$ instead of T we obtain $H_\varphi(t) = H_\varphi^{CRR}(t)$ for $t = 0, ..., T-2$.

4.2 Hedging in a generalized model

In this subsection we assume a general Ω which doesn't have to be finite. Moreover, from now on we assume that the stock price dynamics instead of (2.1) satisfies the following weaker assumption:

Assumption 4.8. $P(\eta_{t+1} < a + \varepsilon) > 0$ *and* $P(\eta_{t+1} > b - \varepsilon) > 0$ *for all* $\varepsilon > 0$ *and* $t = 0, ..., T-1$.

For all $\varepsilon > 0$ and $t = 0, ..., T-1$ let Δ_t^ε denote a set of all sequences of real numbers $\{\delta_n\}_{n=1,...,T-t}$ such that $0 < \delta_n < \varepsilon$ for $n = 1, ..., T-t$.
For all $\varepsilon > 0, t = 0, ..., T-1$ and $\delta \in \Delta_t^\varepsilon$ let $H_{f_\varphi}^{\varepsilon,\delta}(t)$ be defined in the same way as $H_\varphi(t)$ assuming in addition that $P(\eta_{u+1} = a + \delta_{u-t+1}) + P(\eta_{t+1} = b - \delta_{u-t+1}) = 1$ and $0 < P(\eta_{u+1} = b - \delta_{u-t+1}) < 1$ for $u = t, ..., T-1$.
For every $t = 0, ..., T-1$ we have:

Lemma 4.9. If *$f_{\varphi,1}$ and $f_{\varphi,2}$ are continuous then $H_\varphi(t) \subseteq H_\varphi^{CRR}(t)$ for all $\omega \in \Omega$.*

Proof. Let $\omega \in \Omega$ be fixed in this proof.

Assume that $(u, v) \in H_\varphi(t)$. From Assumption 4.8 it is not difficult to show that for all $\varepsilon > 0$ there exists a sequence $\delta \in \Delta_t^\varepsilon$ such that $(u, v) \in H_\varphi^{\varepsilon,\delta}(t)$. Since we can take ε arbitrarily close to 0 and $f_{\varphi,1}, f_{\varphi,2}$ are continuous we get $(u, v) \in H_\varphi^{CRR}(t)$.

Consequently, $H_\varphi(t) \subseteq H_\varphi^{CRR}(t)$ and the proof is therefore completed. □

In our generalized model, we have the following theorem which is similar to Theorem 4.2:

Theorem 4.10. *Let φ be an option such that $f_\varphi \in \Psi$ and $f_{\varphi,1}, f_{\varphi,2}$ are continuous functions. Then $H_\varphi(t) = H_\varphi^{CRR}(t)$ for all $\omega \in \Omega$ and $t = 0, ..., T-1$.*

Proof. It is clear that $H_\varphi(T) = C_{(f_{\varphi,1}^{(0)}(S_T), f_{\varphi,2}^{(0)}(S_T))}$.

Assume that for some $t = 1, ..., T-1$ we have:

$$H_\varphi(t+1) = C_{(f_{\varphi,1}^{(T-t-1)}(S_{t+1}), f_{\varphi,2}^{(T-t-1)}(S_{t+1}))}.$$

Following the lines of the proof of Theorem 4.2 we get $H_\varphi^{CRR}(t) \subseteq H_\varphi(t)$ for all $\omega \in \Omega$. From Lemma 4.9 we have $H_\varphi(t) \subseteq H_\varphi^{CRR}(t)$ for all $\omega \in \Omega$. In consequence, we obtain $H_\varphi(t) = H_\varphi^{CRR}(t)$.

By backward induction we get $H_\varphi(t) = H_\varphi^{CRR}(t)$, for every $t = 0, ..., T-1$ which completes the proof. □

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